INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 39, Northern Spring 2018 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Aladdin has several gold coins and from time to time he asks the Genie to give him more. On each such occasion the Genie first responds by adding a thousand gold coins and then he takes back a half of the total weight of all Aladdin's gold coins. If after asking the Genie for more gold ten times, is it possible for Aladdin that the number of his gold coins has increased taking into account that each time the Genie takes a half of all Aladdin's gold back and no coin is broken into smaller pieces? (4 points)

2. Do there exist 2018 positive reduced fractions, each with a different denominator, such that the denominator of the difference of any two (after reducing to lowest terms) is less than the denominator of any of the initial 2018 fractions? (5 points)

3. One hundred different numbers are written in the squares of a 10×10 table, one number in each square. For each move one can select a rectangle consisting of some squares, and for each square of that rectangle swap its number with the number in the square opposite to it with respect to the centre of the rectangle (i.e. make a rotation of the rectangle by $180°$). Is it always possible to arrange the numbers in the table taking no more than 99 moves so that the numbers increase from left to right in each row, and from bottom to top in each column? (6 points)

4. An equilateral triangle lying in the plane α is orthogonally projected onto a plane β, which is not parallel to α. The resulting triangle is again orthogonally projected onto a plane γ, and its image is an equilateral triangle again. Prove that

   (a) the angle between the planes α and β is equal to the angle between the planes β and γ. (4 points)

   (b) the plane β intersects the planes α and γ along the lines which are perpendicular to each other. (4 points)

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5. You are travelling to some country and you don't know its language. You know that symbols "!" and "?" stand for addition and subtraction, but you don't know which symbol is for which operation. Each of these two symbols can be written between two arguments, but for subtraction you don't know if the left argument is subtracted from the right or vice versa. For example, $a?b$ could mean any of $a - b$, $b - a$ and $a + b$. You don't know how to write any numbers, but variables and brackets can be used as usual. Given two arguments a and b how can you write for sure an expression that is equal to $20a - 18b$? (10 points)

6. Let quadrilateral $ABCD$ be inscribed into a circle S. Let P be the intersection point of the rays BA and CD. Let U and V be the intersection points of the line going through P and parallel to the tangent to S at point D, with the tangents to S at points A and B respectively. Prove that the circumcircle of triangle CUV is tangent to the circle S. (10 points)

7. The King decides to reward a group of n wizards. The wizards are placed in line one after another (so that they can see in the same direction only), each of them wearing either a black or white hat. Each wizard can see the hats of all the wizards in front of him. Starting from the back of the line, each wizard in turn announces a colour (black or white) and a natural number of his choice. The King then counts the number of wizards who nominated the colour of his own hat, and then grants a pay bonus for the same number of days to all the wizards. The wizards are allowed to decide on a common strategy prior to forming the line, but they know that k of them are insane. However, they do not know who of them is insane. An insane wizard tells a white or black colour and a natural number regardless of the common strategy. What is the maximum number of days with bonus pay that can be guaranteed for sure with the common strategy no matter where the insane wizards are placed in the line? (12 points)